

Uhlenbeck compactification as a functor.

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1 Introduction.

We work with noetherian schemes over an algebraically closed field k of arbitrary characteristic. The goal of this paper is to give a functorial definition of a compactification for moduli of vector bundles on a smooth projective surface. More precisely, let $\pi : X \rightarrow S$ be a smooth projective morphism of relative dimension 2 (although the theory can probably be generalized to the Cohen-Macaulay case). We define the *Uhlenbeck moduli functor*, or the functor of *quasibundles* $QBun(r, d)$ that compactifies the stack $Bun(r, d)$ of vector bundles of fixed rank r and $c_2 = d$ along the fibers of π .

It is well known, cf. [DK], that for $k = \mathbb{C}$, $S = Spec(\mathbb{C})$ one can use the Uhlenbeck compactness theorem to show that a family of vector bundles E_t , $u \in \mathbb{C} \setminus \{0\}$ with $c_2 = d$ degenerates, in some sense, to a vector bundle E_0 with a different second Chern class, $c_2(E_0) = d - k$ “plus” an effective 0-cycle ξ of degree k in X , e.g. cf. [DK]. In the original framework of the Uhlenbeck’s Compactness Theorem, the cycle ξ describes a non-negative combination of delta functions which split off the Chern form in the limit, for a family of connections on E_t . Understanding the moduli space $M(r, d)$ in the sense of semistable bundles, one gets a compact Uhlenbeck moduli space $M^U(r, d)$ which has a set theoretic decomposition

$$M^U(r, d) = \coprod_{k \geq 0} M(r, d - k) \times Sym^k X$$

and contains $M(r, d)$ as an open subset, although not necessarily dense. Here Sym^k is the k -th symmetric power of X , i.e. the quotient of the k -fold symmetric product by the action of the symmetric group. In fact, by Bogomolov-Miyaoka inequality only finitely many pieces of this union are nonempty.

In this paper, however, we would like work over an arbitrary algebraically closed field k and consider the moduli functor $QBun(r, d)$, containing rank r vector bundles $Bun(r, d)$ as an open subfunctor, with a certain compactness property (but non-separated - as is the case with $Bun(r, d)$ itself).

A conceptual difficulty here is in treating formal sums $c_2(E) + \xi$ as a single object. The approach adopted in this paper is that both $c_2(E)$ and ξ define multiplicative functors $Pic(X) \rightarrow Pic(S)$ between the groupoid categories of line bundles and their isomorphisms.

For $c_2(E)$ the corresponding functor is a categorical version of the map on cohomology $H^2(X, \mathbb{Z}) \rightarrow H^2(S, \mathbb{Z})$ or the Chow groups $A^1(X) \rightarrow A^1(S)$, obtained by taking the cup product of c_1 for a line bundle L , with $c_2(E)$ and then integrating the class along the fibers of $\pi : X \rightarrow S$. A fancier definition of the functor $\mathfrak{c}_2^E : Pic(X) \rightarrow Pic(S)$, which the author has learned from A. Beilinson, is that an \mathcal{O}^* -torsor L^* on X can be multiplied by the gerbe over the Zariski sheaf \mathcal{K}_2 of algebraic K -groups, corresponding to the class $c_2(E) \in H^2(X, \mathcal{K}_2)$; and the resulting 2-gerbe over \mathcal{K}_3 may be integrated along the fibers of π to give a torsor over \mathcal{O}^* on S . Although in the last chapter we do make some comments on the gerbe approach, the definition used in this paper for vector bundles is that of $\mathfrak{c}_2^E(L)$ as the determinant of

$$R^\bullet \pi_* (L^\vee \rightarrow \mathcal{O}_X) \otimes_{\mathcal{O}_X} (\mathcal{O}_X^{r-1} \rightarrow E \rightarrow \det(E)).$$

We can assume at first that both factors have zero arrows, but in fact we can make them nontrivial by choosing a section of L and $(r-1)$ sections of E - which is an important feature of this construction. The second factor should be thought of as a categorical representative of $c_2(E)$ since the leading term of its Chern character is $c_2(E)$.

On the other hand, in characteristic zero a family of zero cycles ξ may be viewed as a map $\xi : S \rightarrow Sym^d(X/S)$ to the relative symmetric power of X . For fields of arbitrary characteristic one needs to work with similar spaces $\Gamma^d(X/S)$ obtained via the divided powers algebras, cf. [Ro], [Ry]. For a line bundle L on X one has a natural line bundle $\Gamma^d(L)$ on $\Gamma^d(X/S)$ (informally, its fiber is the tensor product of the fibers of the original L , over the d points of the zero cycle). Thus, for any section $\xi : S \rightarrow \Gamma^d(X/S)$ we can set $N_\xi(L) := \xi^*(\Gamma^d(L))$ which gives a functor $Pic(X) \rightarrow Pic(S)$.

Both functors are *multiplicative*, i.e. they agree with tensor products of line bundles in a natural sense. Tensor product of such functors will correspond to the formal sum $c_2(E_0) + \xi$.

The paper is organized as follows. First we define in Section 2 multiplicative functors between Picard categories (in fact we deal with a functor $N : PIC(X/S) \rightarrow PIC(S)$ of larger categories that take base changes into account); and give some examples.

In Section 3 we consider a quadruple (Z, E, N, D) consisting of a vector bundle E on the open complement to a closed subset $Z \subset X$ which is finite over X , an extension of $\det(E)$ to a line bundle D on X and a multiplicative functor N as before. In such a setting we define the notion of an E -localization of N at Z which says, informally, that “on the complement to Z the functor N is identified with \mathfrak{c}_2^E ”. This is an additional structure, as the same N can be localized at different closed subsets Z . Finally, in Section 4 we define a notion of an “effective E -localization”, which boils down to saying that some apriori rational section of a line bundle is actually regular. We need the concept to end up with a functor that eventually has a decomposition with pieces like $M(r, d-k) \times Sym^k X$ before: without localization of

functors we would only be able to work with rational equivalence classes of cycles, and without effectiveness - only with all cycles of fixed degree k rather than effective ones.

In Section 5 we define the Uhlenbeck functor as the functor of quadruples (Z, E, N, D) equipped with an effective E -localization of N at Z . Such objects will be called *quasibundles* in this paper. Although the use of this term does not agree with *some* earlier papers in the field (see below) where “quasibundles” stand for slightly different objects and the corresponding moduli spaces (functors) only map to the one defined here, we still choose this term due to the relation to quasimaps. To explain it briefly, suppose that we have a morphism of sheaves $\mathcal{O}^r \rightarrow E$ which is an isomorphism generically on each fiber over S , and in fact on the complement to a relative curve Y given by the determinant of $\mathcal{O}^r \rightarrow E$. Then by Cramer’s Rule E^\vee is sandwiched between two sheaves

$$\mathcal{O}_X^r(-Y) \subset E^\vee \subset \mathcal{O}_X^r$$

so it can be described through a quotient of the sheaf \mathcal{O}_Y^r . Further, let $(C_t, t \in \mathbb{P}^1)$ be a pencil of curves and assume that none of the C_t shares an irreducible component with Y . Then for any t we get a point in the Quot scheme $Quot_{C_t}(\mathcal{O}_{Y \cap C_t}^r, l)$ parameterizing all quotients of $\mathcal{O}_{Y \cap C_t}^r$ with fixed length l . Note that the scheme-theoretic intersection $Y \cap C_t$ is finite so the Quot scheme naturally embeds as a closed subscheme in a certain product of Grassmannians, and its determinant bundle is the product of the standard determinant bundles restricted from the Grassmannians. Varying t we get a map (i.e. a section) from \mathbb{P}^1 to the corresponding relative Quot scheme, from which the original bundle E can be recovered.

For quasibundles this will be replaced by a quasimap in the sense of [FGK]. This single quasimap does not determine the quasibundle anymore: for a pair (E, ξ) consisting of a bundle and a zero cycle, we can only recover the direct image of ξ to \mathbb{P}^1 (this makes sense if the base locus of the pencil avoids the support of ξ). Still, working with a family of pencils, we can recover a quasibundle from the corresponding family of quasimaps (this is explained in Section 4.5 to clarify the meaning of effective localization).

In Section 5.2 we show that for $S = Spec(k)$ the set of k -points of the stack of quasibundles has the same decomposition as in the case of differential geometric moduli space. We also prove that $QBun(r, n)$ is complete in a certain sense (i.e. establish the existence part in the valuative criterion of properness, but not the uniqueness which is not expected at all in this setting). At the moment we cannot prove that the Uhlenbeck stack is algebraic but Section 5.3 gives a conjectured local flat covering by a scheme of quasimaps into a relative punctual Quot scheme. We also show that the Gieseker functor of flat torsion free families of sheaves maps to the Uhlenbeck functor.

Finally, in Section 6 we explain how to define a similar functor for torsors over split semisimple simply connected groups (the approach may be generalized to the reductive case). One

important modification is that effective zero cycles should be replaced by cycles with coefficients in a semigroup. Then symmetric/divided powers of X will be replaced by their iterated products, as it should be if we expect that

$$QBun(G_1 \times G_2) = QBun(G_1) \times QBun(G_2).$$

In fact, we indicate how to define quasibundles in arbitrary dimension but our treatment is necessarily sketchy since this involves effective codimension 2 cycles, for which a functorial definition in arbitrary characteristic has not been written down yet.

We would like to place our work in regards to other papers on compactifications for the moduli of bundles. The original topological construction of Donaldson in [DK] was reinterpreted in algebro-geometric terms independently by J. Li and J. Morgan, cf. [Li] and [Mo]. A framed version was recently constructed by U. Bruzzo, D. Markushevich and A. Tikhomirov in [BMT]. The general construction of V. Balaji in [Ba1] gives a compactified moduli scheme of principal G -bundles of a surface over \mathbb{C} , which is obtained as a schematic image of another moduli scheme under a certain morphism. This scheme a priori depends on a choice of a representation $\rho : G \rightarrow GL(V)$ (although different choices gives homeomorphic spaces).

In all of these cases the moduli scheme is defined as an image of another scheme and it is not clear from the definition what kind of objects it parameterizes. In a sense, this is the issue addressed in our paper (with appropriate definition of semistability the moduli space of semistable quasibundles - in our sense - is expected to be the scheme constructed in [Ba1]). In addition, the approach of the present paper works in arbitrary characteristic, and can be extended to arbitrary dimension.

In a different direction, a construction of the Uhlenbeck stack briefly outlined by V. Drinfeld in [Dr] has been implemented by A. Braverman, M. Finkelberg, D. Gaitsgory and A. Kuznetsov in [FGK] and [BFG] to yield an appropriate stack of framed quasibundles for the case of projective plane. In a sense, the present paper follows a similar approach, although in a rather different disguise provided by Deligne's theory of intersection bundles, cf. [De], [Du], [El], [MG]. It is rather likely that our definition reduces in the case of projective plane to one of the stacks defined in [FGK] and [BFG]. Ideally, this approach should be phrased in terms of relative K-theory for flat surjective morphisms, but in this area there has not been much progress since [Gr1] and [Gr2] (more specifically, one needs the analogue of global residue homomorphism in K-theory, and something like a relative Gersten resolution). See, however, [Dr] and [OZ] for the techniques on which such relative K-theory could be based.

Other compactifications (or rather their stack versions) admit a natural "generalized blow-down" morphism to our functor of quasibundles. These includes the Gieseker compactification by torsion free sheaves, as long as compactifications by Gómez-Sols in [GS] and A. Schmitt in [Sch], the algebraic version of the bubble-tree compactification by D. Markushevich, A. Tikhomirov and G. Trautmann in [MTT] and a somewhat similar compactification by N. Timofeeva in [Ti]. All of these compactifications remember more than just the multiplicities

of the cycle ξ and consequently none of such moduli functors satisfies the product formula $\mathcal{M}(G_1 \times G_2) \simeq \mathcal{M}(G_1) \times \mathcal{M}(G_2)$.

Finally, a word on terminology. Our use of the word “quasibundles” agrees with that of [FGK] and [BFG] (since ultimately they are linked to quasimaps) and differs from that of [Ba1], [Sch] and [GS].

All unlabeled tensor products are tensor products of quasicoherent sheaves on a scheme over its structure sheaf.

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2 Multiplicative functors

For a morphism $\pi : X \rightarrow S$ we denote by $PIC(X/S)$ the category formed by pairs (T, L) where T is a scheme over S and L is a line bundle over $X_T = X \times_S T$. A morphism $(T_1, L_1) \rightarrow (T_2, L_2)$ is an S -morphism $T_1 \rightarrow T_2$ plus an isomorphism of L_1 with the pullback of L_2 to X_{T_1} . When $X = S$ and π is the identity map, we denote $PIC(X/S)$ by $PIC(S)$.

We choose and fix a locally constant function $d : S \rightarrow \mathbb{Z}$, such as the second Chern class for a family of bundles with base S (when π is a family of smooth projective surfaces); or the degree of a family of zero cycles with base S .

Definition. By a degree d *multiplicative functor* we will understand the data consisting of

- (a) A functor $N : PIC(X/S) \rightarrow PIC(S)$;
- (b) A family of isomorphisms $b_{L_1, L_2} : N(L_1 \otimes L_2) \rightarrow N(L_1) \otimes N(L_2)$ which agree with the associativity and commutativity for tensor products of line bundles; in the sense of Section 1.2.1 of [Du].
- (c) A family of isomorphisms $c_{(T, L)} : N(\pi_T^* L) \simeq L^{\otimes d}$, where L is a line bundle on $T \rightarrow S$ and $\pi_T : X_T \rightarrow T$ is the base change. This should agree with isomorphism of (b) in a natural way, and we also require that an isomorphism $\sigma : L_1 \rightarrow L_2$ corresponds to $c_\sigma = \sigma^{\otimes d} : L_1^{\otimes d} \rightarrow L_2^{\otimes d}$.

We now give three basic examples of multiplicative functors

2.1 Zero Cycles and Families of Artin Sheaves

In characteristic zero, let $Sym^d(X/S) \rightarrow S$ be the relative symmetric power obtained by taking the quotient of the d -fold relative cartesian product of X over S , by the action of the symmetric group Σ_d . If π is projective, so is $Sym^d(X/S) \rightarrow S$. In characteristic $p > 0$ one needs to be more careful and work with divided powers, cf. [Ro], [Ry], which leads to a similar space $\Gamma^d(X/S) \rightarrow S$. For any line bundle L on X one has an induced line bundle $\Gamma^d(L)$ on

$\Gamma^d(X/S)$; in characteristic zero this is simply the descent of $L \boxtimes \dots \boxtimes L$ with respect to the symmetric group action.

A family of zero cycles of degree d with base S is represented by a section $\xi : S \rightarrow \Gamma^d(X/S)$ and therefore we have a functor $N : PIC(X/S) \rightarrow PIC(S)$ which sends $(T \rightarrow S, L)$ to a line bundle $\xi_T^*(\Gamma^d L)$ on T . Here $\xi_T : T \rightarrow \Gamma^d(X/S)_T \simeq \Gamma^d(X_T/T)$ is the family of zero cycles obtained by base change. It is not too difficult to construct the additional data required in the definition of a multiplicative functor, see [Ba2].

A family of zero cycles can be induced by a sheaf \mathcal{A} on X which is flat with finite support over S (i.e. a family of Artin sheaves with base S). In this case ξ simply gives points in the support of \mathcal{A} with multiplicities. Then the multiplicative functor can also be defined by

$$N_{\mathcal{A}}(L) = \det \pi_* [(L^\vee \rightarrow \mathcal{O}_X) \otimes \mathcal{A}]$$

See Chapter 6 of [De], Section 1.2.1 of [Du] for more details. By Section 5 in [Ba2] this is the functorial version of the Hilbert-to-Chow map in the case when \mathcal{A} is a family of zero dimensional subschemes.

2.2 Intersection Bundles

Another example of a multiplicative functor $PIC(X/S) \rightarrow PIC(S)$ is the two-dimensional version of Deligne's intersection bundles introduced in [De]. For three bundles L_0, L_1, L_2 on X we set

$$\langle L_0, L_2, L_2 \rangle := \det R\pi_* [(L_0^\vee \rightarrow \mathcal{O}_X) \otimes (L_1^\vee \rightarrow \mathcal{O}_X) \otimes (L_2^\vee \rightarrow \mathcal{O}_X)]$$

The properties of these construction were studied extensively in [El], [MG], [Du]. Fixing L_1 and L_2 one obtains a multiplicative functor

$$IB_{L_1, L_2} : PIC(X/S) \rightarrow PIC(S); \quad L_0 \mapsto \langle L_0, L_1, L_2 \rangle.$$

2.3 Second Chern Class

In this subsection we give two different constructions for the second Chern class functor $\mathbf{c}_2^E : PIC(X/S) \rightarrow PIC(S)$ of a bundle E on X , and briefly explain why they are equivalent. The second construction is a special case of a more general definition given earlier by R. Elkik; and the first one (which seems to be new) is a bit more adjusted to the concept of a localized multiplicative functor, that we introduce below.

Construction 1. The starting point here is a geometric interpretation of the second Chern class on a smooth projective X . If E is generated by global sections then the rank of a generic morphism $\xi : \mathcal{O}_X^{r-1} \rightarrow E$ will be $\leq (r-2)$ on a codimension 2 closed subset Z_ξ which corresponds to the class $c_2(E)$ in an appropriate version of cohomology group (e.g. Chow

groups or integral cohomology if $k = \mathbb{C}$) - of course, if one takes into account multiplicities. We can restate this using a particular case of the *Buchsbaum-Rim complex*, cf. [BR]. Let $\xi : F \rightarrow E$ be a morphism of vector bundles of ranks $(r - 1)$ and r , respectively. Consider the *dual* Buchsbaum-Rim complex \mathcal{BR}_ξ

$$0 \rightarrow F \rightarrow E \rightarrow \det E \otimes \det F^\vee \rightarrow 0 \quad (1)$$

where the first nonzero arrow is ξ , and the second is obtained from the map $E \otimes \Lambda^{r-1} F \rightarrow \Lambda^r E$ sending $e \otimes \omega$ to $e \wedge \Lambda^{r-1}(\xi)(\omega)$. We will assume that the term $\det E \otimes \det F^\vee$ is placed in homological degree zero. A simple argument shows that this complex is exact away from Z_ξ and has Chern character

$$ch(\mathcal{BR}_\xi) = c_2(E) - c_2(F) + c_1(F)^2 - c_1(F)c_1(E) + \text{higher order terms} \quad (2)$$

When $F = \mathcal{O}_X^{r-1}$ this reduces to $(c_2(E) + \text{higher order terms})$.

Proposition 1 *Suppose that $\pi : X \rightarrow S$ is a flat family of Cohen Macaulay projective surfaces and the degeneracy locus Z_ξ of $\xi : \mathcal{O}_X^{r-1} \rightarrow E$ is finite over S . Then \mathcal{BR}_ξ has non-trivial cohomology in degree zero only, and the sheaf $\mathcal{H}^0(\mathcal{BR}_\xi)$ is flat over S .*

Proof. By definition \mathcal{BR}_ξ is dual to the Buchsbaum-Rim complex, cf. [BR], of the adjoint morphism $\xi^* : E^\vee \rightarrow \mathcal{O}_X^{r-1}$. Since π is Cohen-Macaulay and Z_ξ is finite, for any point $x \in Z_\xi$ and any sheaf \mathcal{E} on S , there exists a length two $\pi^*\mathcal{E}$ -regular sequence in \mathfrak{m}_x . By Theorem 1 of [BR], for any such \mathcal{E} the complex $\mathcal{BR}_\xi \otimes \pi^*\mathcal{E}$ can only have cohomology in degree zero. For $\mathcal{E} = \mathcal{O}_S$ we obtain the statement about the cohomology of \mathcal{BR}_ξ . Viewing \mathcal{BR}_ξ as a resolution of $\mathcal{H}^0(\mathcal{BR}_\xi)$ we conclude also that $\mathcal{T}or_i^{\mathcal{O}_S}(\mathcal{H}^0(\mathcal{BR}_\xi), \mathcal{E}) = 0$ for $i > 0$, i.e. $\mathcal{H}^0(\mathcal{BR}_\xi)$ is flat over S . \square

We can now define a functor

$$\mathfrak{c}_2^E : PIC(X/S) \rightarrow PIC(S)$$

as the functor $N_{\mathcal{A}}$ defined for the family of Artin sheaves $\mathcal{A} = \mathcal{H}^0(\mathcal{BR}_\xi)$ as in Section 2.1. To show that it depends only on E but not ξ and also to define it for bundles which are not generated by global sections, observe that by the Euler isomorphism, cf. e.g. [GKZ], $N_{\mathcal{A}}$ is isomorphic to the functor

$$L \mapsto \det R\pi_{S*}[(L^\vee \rightarrow \mathcal{O}_X) \otimes (\mathcal{O}_X^{r-1} \rightarrow E \rightarrow \det E)] \quad (3)$$

where the arrows in the two tensor factors are zero.

Remark. More generally, it is natural to consider a situation when we have a rank $(r - 1)$ vector bundle G on S and a coherent sheaf morphism $\xi : W = \pi^*G \rightarrow E$ which is an

embedding of vector bundles away from a closed subset Z_ξ , finite over S . This leads to a complex

$$0 \rightarrow \pi^* G \rightarrow E \rightarrow \det E \otimes \det \pi^* G^\vee \rightarrow 0$$

An easy application of the projection formula shows that

$$\det R\pi_*[(L^\vee \rightarrow \mathcal{O}_X) \otimes \mathcal{BR}_\xi] \simeq \mathfrak{c}_2^E(L) \otimes \det G^{-L \cdot \det(E)};$$

where for two line bundles L, D we denote

$$L \cdot D = \chi(L \otimes D) - \chi(L) - \chi(D) + \chi(\mathcal{O}_X)$$

and χ is the Euler characteristic of a vector bundle restricted to fibers of π . By flatness of π this is a locally constant function $S \rightarrow \mathbb{Z}$.

Construction 2. A general construction of R. Elkik in [El] specializes to our case as follows. First consider the projective bundle $\psi : \mathbb{P} := \mathbb{P}(E^\vee) \rightarrow X$ of lines in E , and the corresponding Segre class functor $\mathfrak{s}_2^E : PIC(X/S) \rightarrow PIC(S)$ given by

$$\mathfrak{s}_2^E(L) = \det R\psi_*[(\psi^* L^\vee \rightarrow \mathcal{O}_{\mathbb{P}}) \otimes (\mathcal{O}_{\mathbb{P}}(-1) \rightarrow \mathcal{O}_{\mathbb{P}})^{\otimes(r+1)}].$$

Next, imitating the relation between Chern and Segre classes $c_2(E) + s_2(E) = c_1^2(E)$ we set

$$\mathfrak{c}_2^E(L) := IB_{\det(E), \det(E)}(L) \otimes \mathfrak{s}_2^E(L)^\vee$$

Let us explain briefly why this functor is isomorphic to that of Construction 1. First, one uses the fact that \mathfrak{s}_2^E is multiplicative, cf. [El], to show that the second definition is equivalent to

$$L \mapsto \det R(\pi\psi)_*[(\psi^* L^\vee \rightarrow \mathcal{O}_{\mathbb{P}}) \otimes (\mathcal{O}_{\mathbb{P}}(-1) \rightarrow \mathcal{O}_{\mathbb{P}})^{\otimes r} \otimes (\mathcal{O}_{\mathbb{P}} \rightarrow \mathcal{O}_{\mathbb{P}}(1))].$$

The projection formula for the morphism ψ shows that \mathfrak{s}_2^E is isomorphic to the functor

$$L \mapsto \det R\pi_*[(L^\vee \rightarrow \mathcal{O}_X) \otimes (\det(E)^\vee \rightarrow \mathcal{O}^{r+1} \rightarrow E)]$$

(as before, all arrows in the factors on the right hand side are zero). Now the isomorphism of the two definitions of \mathfrak{c}_2^E follows from a similar isomorphism

$$\begin{aligned} IB_{\det(E), \det(E)}(L) &\simeq \det R\pi_*[(L^\vee \rightarrow \mathcal{O}_X) \otimes (\det(E)^\vee \rightarrow \mathcal{O}_X) \otimes (\mathcal{O}_X \rightarrow \det(E))] \\ &\simeq \det R\pi_*(L^\vee \rightarrow \mathcal{O}_X) \otimes (\det(E)^\vee \rightarrow \mathcal{O} \oplus \mathcal{O} \rightarrow \det(E)) \end{aligned}$$

Thus, the streamlined version of Construction 2, is given by:

$$\mathfrak{c}_2^E(L) := IB_{\det(E), \det(E)}(L) \otimes \mathfrak{s}_2^E(L)^\vee$$

where

$$\mathfrak{s}_2^E(L) \simeq \det R\pi_*(L^\vee \rightarrow \mathcal{O}_X) \otimes (\det(E)^\vee \rightarrow \mathcal{O}^{r+1} \rightarrow E)$$

Comparing the definitions we see that the above formula gives the same functor as Construction 1.

3 Localization of multiplicative functors

Let $Z \subset X$ be a closed subset, finite over S . Consider a bundle E on the open subscheme $U = X \setminus Z$. To be able to say that a multiplicative functor $N : PIC(X/S) \rightarrow PIC(S)$ “restricts to \mathfrak{c}_2^E over U ” we need the concept of an E -localization of N at Z . By definition, this is additional data described as follows.

Consider a morphism $\phi : L_1 \rightarrow L_2$ of line bundles on X viewed as coherent sheaves, and assume that ϕ is an isomorphism along Z . Then $Coker(\phi) \otimes (\mathcal{O}^{r-1} \rightarrow E \rightarrow \det(E))$ is a well-defined perfect complex on X : although the second factor is only defined on U , the support of the first factor is contained in U . To simplify notation we will assume $L_2 \simeq \mathcal{O}_X$ but the general case can be derived easily.

Definition. An E -localization of N at Z consists of a family of isomorphisms

$$a_\phi : \det \pi_* [\mathcal{O}_Y \otimes (\mathcal{O}^{r-1} \rightarrow E \rightarrow \det(E))] \simeq N(L),$$

for any $\phi : L^\vee \rightarrow \mathcal{O}$ which vanishes on a relative divisor Y disjoint from Z . We require that such isomorphisms should agree with tensor product on sections and values of N .

We have the following basic examples

Supports of zero cycles. If $N : PIC(X/S) \rightarrow S$ is the functor describing the family of zero cycles, then N carries a canonical \mathcal{O} -localization at the closed subset Z_ξ which is the support of the relative zero cycle (or the support of the family of Artin sheaves). We remark here that a rigorous definition of the support for a family $\xi : S \rightarrow \Gamma^d(X/S)$ of zero cycles, involves the addition map $add : X \times_S \Gamma^{d-1}(X/S) \rightarrow \Gamma^d(X/S)$, cf. [Ry], [Ba2]. Then the support Z_ξ is simply the projection of the closed subset $add^{-1}(\xi(S))$ to X .

Second Chern class. For a bundle E on X the definition of $N = \mathfrak{c}_2^E$ implies that this multiplicative functor has a canonical E -localization at $Z = \emptyset$.

Generic morphism $\mathcal{O}^{r-1} \rightarrow E$. Suppose we have a morphism $\phi : \mathcal{O}^{r-1} \rightarrow E$ which has rank $\leq (r-2)$ on a closed subset $Z \subset X$. Then the non-zero arrows of the dual Buchsbaum-Rim complex $\mathcal{O}^{r-1} \rightarrow E \rightarrow \det(E)$ make it quasiisomorphic to a family of Artin sheaves \mathcal{A} supported at Z , hence this induces an \mathcal{O} -localization of \mathfrak{c}_2^E at Z .

The last example illustrates that an E -localization is really some additional data which cannot be recovered from N alone - different choices of ϕ will in general lead to different closed subsets Z . Hence the same functor N may admit localizations at different closed subsets. This is somewhat similar to the fact that an effective divisor on a curve defines a line bundle but different effective divisors may give isomorphic line bundles.

Lemma 2 *Any functor automorphism of c , which agrees with the multiplicative structure and the localization on Z , is trivial.*

Proof. By definition, a functor automorphism of c should send a pair (L, T) to an automorphism of the line bundle $c(L)$ on T . Agreement with the multiplicative structure of c means that for $T \rightarrow S$ we have a group homomorphism $\eta : \text{Pic}(X_T) \rightarrow \Gamma(\mathcal{O}_T^*)$, from the group $\text{Pic}(X_T)$ of isomorphism classes of line bundles on X_T , which behaves naturally with respect to pullback on S -schemes.

To show that η is trivial on the class of L we may assume that T is affine and L is very ample. Then we can cover T with smaller open subsets T_i and choose over each T_i a section $\phi_i : \mathcal{O}_{X_T} \rightarrow L$ which vanishes away from Z_{T_i} . By agreement with localization, over each T_i we must have $\eta(L) = \eta(\mathcal{O}_{X_T}) = 1$, as required. \square

4 Effectiveness conditions for localizations

Effectiveness is a condition imposed on localizations, and roughly speaking it means that some a priori rational section of a line bundle is actually regular. We first explain how this condition appears in examples, where it is based on the following result, obtained by combining Theorem 3, Proposition 9 and Lemma 1 on p. 51 in [MK] (the notation and formulation is changed to match our situation):

Proposition 3 *Let $\pi : X \rightarrow Y$ be a proper morphism of noetherian schemes of finite Tor-dimension, and assume that Y is normal. If \mathcal{F}^\bullet is a perfect complex on X such that*

1. *for each depth 0 point $y \in Y$, restriction of \mathcal{F}^\bullet to the fiber over y is exact,*
2. *for each depth 1 point $y \in Y$, restriction on \mathcal{F}^\bullet to the fiber over y has cohomology only in degree zero, and this unique cohomology sheaf has 0-dimensional support.*

Then the natural trivialization of $\det R\pi_(\mathcal{F})^\bullet$ over an open subscheme of Y (which exists due to (1)) extends canonically to a regular section $\text{Div}(\mathcal{F}^\bullet)$ of the line bundle $\det R\pi_*(\mathcal{F}^\bullet)$ and for any line bundle L on X one has $\text{Div}(\mathcal{F}^\bullet) = \text{Div}(\mathcal{F}^\bullet \otimes L)$.*

4.1 Zero cycles and generic sections of vector bundles.

Suppose we are in a situation of Section 2.1 when there is a flat family of Artin sheaves \mathcal{A} on X and assume for simplicity in this subsection that $S = \text{Spec}(k)$. Choose a line bundle L on X generated by a subspace H of its global sections and set $P = \mathbb{P}(H)$. Then the bundle $\mathcal{O}_P(1) \boxtimes L$ on $X_P = P \times_k X$ has a canonical section s . If the arrow in the complex $(\mathcal{O}_P(-1) \boxtimes \mathcal{A}(L^\vee) \rightarrow \mathcal{O}_P \boxtimes \mathcal{A})$ on X_P is given by $s^\vee \otimes \text{Id}_{\mathcal{A}}$ then this complex becomes exact over the open subset $U_P \subset P$ of all curves in the linear system which avoid the finite support of \mathcal{A} . The second condition of the previous Proposition is also satisfied. By the result quoted above the trivialization of $\det R\pi^*[\mathcal{O}_P(-1) \boxtimes \mathcal{A}(L^\vee) \rightarrow \mathcal{O}_P \boxtimes \mathcal{A}] \simeq \mathcal{O}_P(d)$ over the open

subset U extends to a regular section Div . In fact, each point $x \in Supp(\mathcal{A})$ corresponds to a hyperplane $H_x \subset Y$ of curves in the linear system that contain x , and the section Div is the product of the linear equations for H_x each repeated m_x times, where m_x is the multiplicity of x in the support cycle of \mathcal{A} .

Now suppose that a morphism $\psi : \mathcal{O}_X^{r-1} \rightarrow E$ which has rank $\leq (r-2)$ at a finite subset Z . Then the functor \mathfrak{c}_2^E acquires an E -localization at Z : we can first replace the zero arrows in $\mathcal{O}_X^{r-1} \rightarrow E \rightarrow \det(E)$ by the arrows of the dual Buchsbaum-Rim complex and then replace the complex itself by its cohomology sheaf \mathcal{A} in degree zero. Again, for a line bundle L the trivialization of $\mathcal{O}_P(d)$ at the open subset of sections vanishing away from Z , extends to a regular section.

We also observe here that if E is generated by global sections then a sufficiently general φ will have rank $\leq (r-2)$ at most at a finite subset of X . Indeed, choose a surjection $W \otimes_k \mathcal{O}_X \rightarrow E$ and let $G = Gr(r-1, W)$. If $\mathcal{Z} \subset G \times X$ is the closed subset of pairs (ψ, x) such that ψ has rank $\leq (r-2)$ at $x \in X$. Then for every $x \in X$ its preimage in G is a standard Schubert variety of codimension 2. Therefore, \mathcal{Z} is irreducible of the same dimension as G . Hence for the projection $\mathcal{Z} \rightarrow G$, generic $\varphi \in G$ has a finite fiber, as required.

Remark. For a line bundle L , let H be a vector subspace of $H^0(X, L)$ which generates L and W as above. Using similar arguments one can construct a similar trivialization of the line bundle $\mathcal{O}_P(d) \boxtimes \mathcal{O}_G(L \cdot \det(E))$ on the open subset of all pairs $(s, \psi) \in P \times G$ for which ψ has maximal rank $(r-1)$ everywhere at the zero set of s ; and also extend it to a *regular* section of $\mathcal{O}_P(d) \boxtimes \mathcal{O}_G(L \cdot \det(E))$. This generalizes the classical Chow form for cycles in a projective space.

4.2 Twists by a line bundle.

A general bundle E will be generated by a finite dimensional space of its sections W only after a twist by an appropriate very ample line bundle (in a relative situation W would be a pullback for some bundle on S). We would like to explain how the functor \mathfrak{c}_2 changes when E is replaced by $E(M)$ for some line bundle M . On the level of cohomology classes, if x_i are the Chern roots of E and h is the first Chern class of M then $x_i + h$ are the Chern roots of $E(M)$. Therefore we have a formula

$$c_2(E(M)) = c_2(E) + (r-1)hc_1(E) + \frac{r(r-1)}{2}h^2.$$

Its categorical version is an isomorphism of functors

$$\mathfrak{c}_2^{E(M)} \simeq \mathfrak{c}_2^E \otimes IB_{M, \det(E)}^{\otimes(r-1)} \otimes IB_{M, M}^{\otimes \binom{r}{2}}.$$

and its proof is an easy exercise left to the interested reader (see the end of [MK] for a general approach). Denoting by $c(\det(E), M)$ the last two factors, we can write it in the form

$$\mathfrak{c}_2^{E(M)} \simeq \mathfrak{c}_2^E \otimes c(\det(E), M).$$

Similarly, for any multiplicative functor $N : PIC(X/S) \rightarrow PIC(S)$ we will write

$$N^{(M)} = N \otimes c(\det(E), M).$$

It is straightforward to check that the twisted cycle has degree

$$d(M) = d + (r - 1)M \cdot \det(E) + \binom{r}{2} M \cdot M.$$

It follows from the definitions that an E -localization of N at Z induces an $E(M)$ localization of $N(M)$ at Z .

4.3 Effective localizations.

Now we return back to the situation when E is defined only on the complement U of a closed $Z \subset X$ which is finite along the fibers over S . We fix a multiplicative functor $N : PIC(X/S) \rightarrow PIC(S)$ and its E -localization at Z , which means that for a section $\phi : \mathcal{O}_X \rightarrow L$ vanishing on a relative curve C disjoint from Z , we are given an isomorphism

$$a_\phi : \det R\pi_* [\mathcal{O}_C \otimes (\mathcal{O}^{r-1} \rightarrow E \rightarrow \det(E))] \simeq N(L),$$

where $\phi^\vee : L^\vee \rightarrow \mathcal{O}_X$ is the adjoint.

After a base change $T \rightarrow S$ and a twist by an ample line bundle M on $X_T = X \times_S T$ we can find a morphism $\psi : \mathcal{O}^{\oplus(r-1)} \rightarrow E_T(M)$ on the open complement U_T to the closed subset Z_T , which has maximal rank $(r - 1)$ away from a closed subset of U_T which is also finite over T . Denoting by Z_ψ the union of this subset with Z_T we have that ψ has maximal rank $(r - 1)$ on the open complement U_ψ to Z_ψ . But then we can choose nonzero arrows in $(\mathcal{O}^{r-1} \rightarrow E_T(M) \rightarrow \det(E_T(M)))$ as in the dual Buchsbaum-Rim complex of ψ and obtain an exact complex on U_ψ . Thus, an E -localization of N and a choice of ψ induce a trivialization of $N^{(M)}(L)$ for any section ϕ of L which is nonzero along Z_ψ , and this trivialization agrees with tensor products on line bundles and their sections.

In other words, we have extended the domain of $N^{(M)}$ to the category $PIC_{Z_\psi}(X_T/T)$ which still has line bundles as objects but now morphisms are coherent sheaf morphisms which are nonzero at every point of Z_ψ . By [Ba3] such a functor must come from a relative zero cycle on X_T , but not necessarily effective (as a difference of two effective cycles will also be trivially localized at the union of their supports). This motivates the following definition:

Definition. An E -localization a of N is *effective* if for any base change $T \rightarrow S$ and a morphism $\psi : \mathcal{O}^{r-1} \rightarrow E_T(M)$ such that M is relatively ample and ψ has maximal rank on the complement of $Z_\psi \subset X_T$ as above, the functor $N^{(M)}$ with its \mathcal{O} -localization at Z_ψ is isomorphic to the functor of an effective zero cycle $\xi : T \rightarrow \Gamma^{d(M)}(X_T/T)$ with support in Z_ψ .

For the readers convenience we recall that this means that the extension

$$N^{(M)} : \text{PIC}_{Z_\psi}(X_T/T) \rightarrow \text{PIC}(T)$$

actually extends further to a functor $\text{PIC}^+(X_T/T) \rightarrow \text{PIC}^+(T)$ where both PIC^+ have the same objects as before, but now morphisms are arbitrary morphisms of coherent sheaves. This is actually a certain condition imposed on the extension to PIC_{Z_ψ} : if we have a family of sections ϕ_t , $t \in \mathbb{A}^1$ for a line bundle L and for $t \neq 0 \in \mathbb{A}^1$ the zero set of ϕ_t avoids Z_ψ , then the pullback of $N(L)$ to $\mathbb{A}^1 \times_k T$ is trivialized on $(\mathbb{A}^1 \setminus 0) \times_k T$ and we require that this trivialization extends to a regular section on $\mathbb{A}^1 \times_k T$.

Below we interpret effectiveness condition in terms of quasimaps to punctual Quot schemes of curves in X .

4.4 Relation to effective zero cycles.

Proposition 4 *Suppose that E is a bundle defined everywhere on X and $N : \text{PIC}(X/S) \rightarrow \text{PIC}(S)$ is a multiplicative functor with an effective E -localization at a closed subset $Z \subset X$ finite over S . Then there exists an effective 0-cycle ξ of degree $k = \deg N - \deg \mathfrak{c}_2^E$ with support on Z and a multiplicative isomorphism of functors that agrees with localizations*

$$N \simeq \mathfrak{c}_2^E \otimes N_\xi$$

Proof. By assumption \mathfrak{c}_2^E is well defined so we can form the multiplicative functor $N_1 = N \otimes (\mathfrak{c}_2^E)^{-1}$. Comparing E -localizations we see that for any section $\phi : \mathcal{O}_X \rightarrow L$ of a line bundle which does not vanish at Z (perhaps chosen after a base change on S), we are given an isomorphism

$$a_\phi : \mathcal{O} \simeq N_1(L);$$

and such isomorphisms behave multiplicatively with respect to tensor products of line bundles and their sections.

We would like to show that a_ϕ can be extended to a coherent sheaf morphism even in the case when ϕ does vanish at the points of Z . Since N_1 does not change after twist by a relatively ample M , and effectiveness can be detected by values on sufficiently ample L , we can localize on S and assume that L and E are generated by finite dimensional spaces H, W of their sections. Set $P = \mathbb{P}(H)$ as before.

Over the open subset $\mathcal{U} \subset S \times_k P$ of sections that do not vanish at Z the bundle $N_1(L \boxtimes \mathcal{O}_P(1)) = N_1(L) \otimes \mathcal{O}_P(k)$ is trivialized by assumption. We need to explain why the trivializing section a_ϕ is regular along $P \setminus \mathcal{U} = Z_P$ (fiber by fiber over S our Z_P is a union of hyperplanes in P formed by curves passing through points of Z). To that end, we can assume $S = \text{Spec}(k)$. Choose a point $p \in Z_P$, which corresponds to a curve $C_p \subset X$. In the Grassmannian G of $(r-1)$ -dimensional subspaces in W , the subscheme of all $g \in G$ which induce a linear map $U_g \rightarrow W \rightarrow E_x$ of non-maximal rank $\leq (r-2)$ is a codimension 2 Schubert cell, for any $x \in C_p$. Therefore we can choose a subspace of sections U_g of dimension $(r-1)$ which has maximal rank everywhere on C_p . Then, in the neighborhood of p we obtain a trivialization of $\mathfrak{c}_2^E(L \otimes \mathcal{O}_P(1))$ and its product with a_ϕ is a regular section of $N(L \otimes \mathcal{O}_P(1))$, by assumption. Hence a_ϕ itself was regular in the neighborhood of p to begin with.

Therefore, the effective localization of N extends N_1 to a multiplicative degree k functor $\text{PIC}(X/S)^+ \rightarrow \text{PIC}(S)^+$ of larger categories which still have line bundles as objects but morphisms are taken in the sense of coherent sheaves. By the main result of [Ba2], such a functor is isomorphic to a functor N_ξ associated to a unique degree k effective cycle ξ . \square

Perhaps this is an appropriate place to explain why just considering bundles E away from finite subsets is not good enough, and also a functor N with an effective localization is needed. If X is a smooth surface over a field then any vector bundle E on a complement U to a finite set $Z \subset X$ extends uniquely to X : its direct image as a coherent sheaf satisfies Serre's S_2 condition and then by Auslander-Buchsbaum formula the direct image should be projective at the points of Z . However, the second Chern class of this extension does not remain constant when we start varying E and Z in a family. Yet we have a semi-continuity statement.

Proposition 5 *Assume that the base S is a Noetherian scheme over k and $Z \subset X$, E are as before. For any point $s \in S$ let F_s be the canonical extension of $E|_{X_s}$ to a locally free sheaf on X_s and set $f(s) = \deg c_2(F_s)$. Then $f : S \rightarrow \mathbb{Z}$ is a lower-semicontinuous function: if s_1 is in the closure of s_2 then $f(s_1) \leq f(s_2)$.*

Proof. We can represent E^\vee as a cokernel of the map $G_1|_U \rightarrow G_0|_U$ where G_i are direct sums of line bundles on X (with ample dual line bundles). Therefore on U we have

$$0 \rightarrow E \rightarrow G_0^\vee \rightarrow G_1^\vee$$

Hence if F is the direct image of E to X then

$$H^0(X, F) = \text{Ker}(H^0(X, G_0^\vee) \rightarrow H^0(X, G_1^\vee))$$

Moreover, the same will hold for any base change $T \rightarrow S$: the cohomology of the direct image extension of E_T is always the kernel of the morphism on global sections of $(G_0)_T^\vee$ and $(G_1)_T^\vee$. The same holds if we twist all sheaves by a relatively ample line bundle L on X .

Since the rank of the morphism $\pi_* G_0^\vee(L) \rightarrow \pi_* G_1^\vee(L)$ is a lower semi-continuous function on S (assuming both sheaves are locally free) we conclude that $h^0(F_s(L|_{X_s}))$ is an upper semi-continuous function on S . For an ample enough L this is equal to the Euler characteristic of $F_s(L|_{X_s})$ which by Riemann-Roch is of the form $A - c_2(F_s)$ where A is independent of s . Therefore $f(s) = c_2(F_s)$ is lower semi-continuous. \square

We see from the previous proposition that just deforming E would give a badly behaved functor: a bundle with $c_2 = n$ may be deformed to a bundle with $c_2 = n + k$ and a priori there is no bound on the non-negative integer k . Formally, this is manifested in the fact that $H^1(U, \text{End}(E))$ has an infinite dimensional piece coming from the local cohomology of $\text{End}(E)$ at $x \in Z$. However if we know that N is the product of \mathfrak{c}_2^E and the functor of an effective cycle of degree k_0 then it follows that $k \leq k_0$. This keeps the tangent vectors of first order deformations within a smaller subspace which is closely related to the standard increasing filtration on the local cohomology $\mathcal{H}_x^2(\text{End}(E))$.

4.5 Relation to (quasi)maps into a Quot scheme.

Let us start with a situation when E is defined everywhere on X , choose a rank r bundle W on X which is trivial on the fibers of π (i.e. W is a pullback of a bundle on S) and suppose that we have an injective morphism of coherent sheaves $\rho : W \rightarrow E$ which is an isomorphism at the generic point of each fiber X_s , $s \in S$. Under this assumption both ρ and its adjoint are embeddings of coherent sheaves on each fiber and the cokernel $K := W^\vee / E^\vee$ is flat a Cohen-Macaulay of pure relative dimension 1 over S . So we can view E^\vee as a kernel of the surjection $W^\vee \rightarrow K$ and using Cramer's Rule we see that K is annihilated by the section $\det(\rho)$ of $\det(E) \otimes \det(W)^\vee$, i.e. K is supported on the zero scheme Y of $\det(\rho)$. Hence E^\vee is the kernel of a composition

$$W^\vee \rightarrow W^\vee|_Y \rightarrow K.$$

Suppose that L is sufficiently ample and set $H = \pi^* \pi_* L$, $P = \mathbb{P}(H)$. We view the incidence subscheme $\mathcal{C} \subset P \times_S X$ as a relative curve $\mathcal{C} \rightarrow P$. Consider the open subset $\mathcal{U} \subset P$ of those curves that have no common irreducible components with Y (if L is ample enough the closed complement to \mathcal{U} will have high codimension). Pull K back to $P \times_S X$ and let $K|_{\mathcal{C}}$ be the restriction to \mathcal{C} . Then $K|_{\mathcal{C}}$ is flat of finite length $l = \det(E) \cdot L$ over \mathcal{U} , since the restriction of E^\vee to the fibers of \mathcal{C} over \mathcal{U} has degree $-l$. For $p \in \mathcal{U}$ and the corresponding curve $C_p \subset X$ we have a quotient map

$$W^\vee|_{Y \cap C_p} \rightarrow K|_{C_p}. \tag{4}$$

The source and target have length rl and l , respectively, over the residue field of p . Let \mathcal{W} be the rank rl vector bundle $\pi_*(W^\vee|_{Y \cap \mathcal{C}})$ on \mathcal{U} . The above construction defines a section $s : \mathcal{U} \rightarrow \text{Gr}(\mathcal{W}|_{\mathcal{U}}, l)$ of the Grassmannian bundle of rank l quotients over \mathcal{U} .

Let $\mathcal{O}_{Gr}(1)$ be the determinant of the universal quotient bundle on the relative Grassmannian and \mathcal{K} its pullback to \mathcal{U} via s . Then s can also be recovered from the induced surjection on \mathcal{U}

$$\Lambda s : \Lambda^l \mathcal{W} \rightarrow \mathcal{K}$$

which satisfies Plücker conditions. If we assume that $\Lambda^l \mathcal{W} \rightarrow \mathcal{K}$ is only generically surjective on fibers of $\mathcal{U} \rightarrow S$ (but still satisfies Plücker conditions), we get a concept of a *quasimap* from \mathcal{U} to the relative Grassmannian.

Usually, the concept of a quasimap is used when P has relative dimension 1 over S , i.e. it is a family of projective lines. In this setting P would be a projectivization not for the full bundle of sections of L , but its rank 2 subbundle. Assume for simplicity that $S = \text{Spec}(k)$ and the 2-dimensional subspace $H \subset H^0(X, L)$ is chosen so that $\mathcal{U} = P \simeq \mathbb{P}^1$ (i.e. we have a pencil of curves none of which shares a component with Y). Then, for a quasimap from P to the Grassmannian $Gr(\mathcal{W}, r)$ the image of the morphism $\Lambda s : \Lambda^l \mathcal{W} \rightarrow \mathcal{K}$ is a rank one locally free subsheaf of \mathcal{K} with quotient supported at some points $p_1, \dots, p_k \in \mathbb{P}^1$ and multiplicities a_1, \dots, a_k . In other words, we can view a quasimap s as the usual map with pullback line bundle $\mathcal{K}(-\sum a_i p_i)$ which is “twisted” by $\sum a_i p_i$, the defect of our quasimap.

The point of this discussion is that, similarly to the case when a bundle E defines a map to the relative Grassmannian, a quadruple (Z, E, N, D) with an effective E -localization defines a quasimap $\mathcal{U} \rightarrow Gr(\mathcal{W}|_{\mathcal{U}}, r)$. The “effectiveness” condition is responsible for the fact that Λs is a regular morphism of coherent sheaves, not rational. The main tool here is the following

Lemma 6 *If E is defined everywhere on X and $d = \deg c_2(E)$ then the line bundle \mathcal{K} on $\mathcal{U} \subset P$ is isomorphic to*

$$\det(\pi_{\mathcal{U}})_*(\mathcal{O}_{Y \cap \mathcal{C}}) \otimes \mathfrak{c}_2^E(L) \otimes \mathcal{O}_P(d)|_{\mathcal{U}}$$

Proof We will pull the bundle \mathcal{K} back to the product $P \times_S G$ where $G = Gr(r-1, \mathcal{W})$ is the Grassmannian of $(r-1)$ -dimensional subspaces in \mathcal{W} , and will show a similar decomposition on that product, which will imply the decomposition on P . The isomorphism to be proved is based on the following diagram of sheaves on $X \times G$ with exact columns (Q being the dual to the universal rank $(r-1)$ sub-bundle on G and $\mathcal{O}_G(1)$ the universal quotient bundle):

$$\begin{array}{ccccccc}
0 & \longrightarrow & \det(E^\vee) \otimes \mathcal{O}_G(-1) & \longrightarrow & \det(W^\vee) \otimes_k \mathcal{O}_G(-1) & \longrightarrow & K \\
& & \downarrow & & \downarrow & & \parallel \\
0 & \longrightarrow & E^\vee & \longrightarrow & W^\vee & \longrightarrow & K \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & Q & \xlongequal{\quad} & Q & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \\
& & \mathcal{A} & \longrightarrow & 0 & &
\end{array}$$

Applying the Snake Lemma to the two middle rows of the diagram, we get a short exact sequence

$$0 \rightarrow \mathcal{O}_Y \boxtimes \mathcal{O}_G(-1) \rightarrow K \rightarrow \mathcal{A} \rightarrow 0.$$

If we now pull this back to $X \times G \times P$, intersect with the preimage of the universal curve $C \subset X \times P$ and then push forward to $G \times P$, this will split the determinant for the direct image of K , into the tensor product of $\Lambda^l \pi_*(\mathcal{O}_{Y \cap C}) \otimes \mathcal{O}_G(-l)$ and $\mathfrak{c}_2^{E^\vee}(L \boxtimes \mathcal{O}_P(1)) \otimes \mathcal{O}_G(l)$. Now the assertion follows from a canonical isomorphism

$$\mathfrak{c}_2^E \simeq \mathfrak{c}_2^{E^\vee}$$

(which is a well-known statement for cohomology classes, and for the multiplicative functors can be derived from Elkik's construction in a straightforward way). \square

This can be rephrased as follows (assume $S = \text{Spec}(k)$, $\mathcal{U} = P$ to simplify things). For p in an open subset $\mathcal{U}_Z \subset P$ of curves avoiding Z we have the same picture as before: a length l quotient K_p of $W^\vee|_{C_p \cap Y}$ inducing a map

$$\mathcal{U}_Z \rightarrow \text{Quot}(W^\vee|_{C_p \cap Y}, l) \subset \text{Gr}(\mathcal{W}, l) \subset \mathbb{P}(\Lambda^l \mathcal{W})^\vee.$$

So on \mathcal{U}_Z we have a surjective morphism of vector bundles

$$\Lambda^l \mathcal{W} \rightarrow \mathcal{K}.$$

By the above, its target \mathcal{K} extends to a line bundle on the whole P as

$$\det R\pi_*(\mathcal{O}_C \otimes^L \mathcal{O}_Y) \otimes N(L) \otimes \mathcal{O}_P(d)$$

and in fact $\Lambda^l \mathcal{W} \rightarrow \mathcal{K}$ extends to P as well, although the extension may not be surjective. This means that the map of \mathcal{U}_Z to the relative Grassmannian $\text{Gr}(\mathcal{W}, l)$ is now extended to a *quasimap* on P , in the sense of [FGK]. This provides a link with [FGK] and [BFG] since our Quot scheme naturally embeds into a locus of the affine Grassmanian of the curve C_p , where the elementary modifications of bundles are performed only at the points of $C_p \cap Y$.

5 The Uhlenbeck (or quasibundle) functor

5.1 Definition.

Let $\pi : X \rightarrow S$ a smooth projective morphism of relative dimension 2, as before. We define the groupoid category $QBun(r, d)$ of degree d rank r *quasibundles* on X over S , as follows. An object of this category consists of a

1. A closed subset $Z \subset X$ which is finite over S and a rank r vector bundle E on the open complement $U = X \setminus Z$;
2. A line bundle D on X with a given isomorphism $D|_U \simeq \det(E)$;
3. A degree d multiplicative functor $N : PIC(X/S) \rightarrow PIC(S)$ with an effective E -localization at Z .

For two such objects, a morphism is given by an isomorphism $\psi : E_1 \rightarrow E_2$ over an open subset $U^\circ \subset U_1 \cap U_2$, with closed complement Z° finite over S , and an isomorphism of functors $N_\psi : N_1 \simeq N_2$, which agrees with multiplicative structures and localizations at Z° .

Any base change $T \rightarrow S$ induces a quasibundle on T : for N extension to base changes is a part of the definition of a multiplicative functor, and for E, Z, D we can apply the natural base change definitions. This easily extends to the case when S is an algebraic space (then, of course we use étale topology on S instead of Zariski), which is a bit more appropriate from the point of view of algebraic stacks.

5.2 Completeness and points over $\text{Spec}(k)$.

Proposition 7 *The functor $Bun(r, d) \subset QBun(r, d)$ of rank r bundles with $c_2 = d$, is open in $QBun(r, d)$.*

It follows from the definitions that $Bun(r, d)$ is formed by those quasibundles for which Z is empty and $N = \mathbf{c}_2^E$.

Proof. Let (Z, E, N, D) be a quasibundle with base S and suppose that after a base change $T \rightarrow U$ the pullback quasibundle (Z_T, E_T, N_T, D_T) is isomorphic to a usual bundle. We need to show that T factors through an open subscheme $S^\circ \subset S$ (possibly empty).

By Section 4.4 for any point $s \in S$ the restriction of E to the fiber X_s extends canonically to a vector bundle F_s with $c_2 = d - k$, $k \geq 0$. Moreover, the subset $S^\circ \subset S$ of all points for which $k = 0$ is open in S . If the base change for a quasibundle under $T \rightarrow S$ is isomorphic to an honest bundle, then E_T extends to a vector bundle with $c_2 = d$ along the fibers. Choosing a point $t \in T$ and restricting to its fiber, we see that the image of t must be in S° , hence the base change morphism in fact factors through $T \rightarrow S^\circ$.

In the opposite direction, let us assume that $S^\circ = S$ and prove that the quasibundle is in fact an honest bundle. By Section 4.4 it suffices to show that the direct image F of E to X is locally free. As in *loc. cit.* after an ample twist of E we have a sequence

$$0 \rightarrow E \rightarrow G_0^\vee|_U \rightarrow G_1^\vee|_U$$

where the two bundles on the left are direct sums of ample line bundles. Our assumption $S^\circ = S$ and the argument of Section 4.4 imply that $\pi_* G_0^\vee \rightarrow \pi_* G_1^\vee$ has constant rank hence

$\pi_* F$ is locally free on S . This also holds for any ample twist of E and thus F is flat over S . Then the direct image extension from U to X . Since on each fiber X_s such an extension F_s is locally free, for $i \geq 1$ and $x \in X_s$ we have $\mathcal{E}xt_{X_s}^i(F|_{X_s}, k(x)) = 0$ where $k(x)$ is the skyscraper sheaf at x . By flatness of F and the change of rings spectral sequence we get that $\mathcal{E}xt_X^i(F, k(x)) = 0$ for any $x \in X$. Hence F is locally free. \square

Remark. In fact, if $S_k \subset S$ is the locally closed subscheme on which $c_2(F_s) = d - k$ and a base change morphism $T \rightarrow S$ factor through S_k then one can show by a similar technique that E_T extends by direct image to a vector bundle F with $c_2 = d - k$ along the fibers and the functor N splits into the functor \mathfrak{c}_2^F and a functor induced by a relative family of degree k cycles on T .

The next result explains in which sense $QBun(r, d)$ is a compactification of $Bun(r, d)$. Recall that for schemes the valuative criterion of properness has two parts: a certain morphism should admit an extension, and such extension should be unique. For functors of bundles uniqueness is too much to ask: a family of bundles over a punctured curve has in general several non-equivalent extensions to the puncture. On the other hand, one may still ask about existence of such an extension.

Proposition 8 *If \mathbb{O} is a discrete valuation ring morphism $\text{Spec}(\mathbb{O}) \rightarrow S$ and \mathbb{K} is its field of fractions then each \mathbb{K} -valued point of $QBun(r, d)$ lifts (non-uniquely!) to a \mathbb{O} -valued point of $QBun(r, d)$.*

Proof. A vector bundle over an open subset of $X_{\mathbb{K}} = X \times_S \text{Spec}(\mathbb{K})$ extends to a vector bundle F on $X_{\mathbb{K}}$ itself by direct image. Hence N will be of the form $\mathfrak{c}_2^F \otimes N_{\xi}$, for an effective cycle ξ of certain degree k . Since $\Gamma^k(X/S)$ is represented by a proper scheme over S , the cycle ξ admits a unique extension over \mathbb{O} . On the other hand, by Langton's Theorem, we can find a torsion free sheaf $F_{\mathbb{O}}$, flat over \mathbb{O} , which restricts to $F_{\mathbb{K}}$ over \mathbb{K} . It will be locally free away from a closed subset of $X_{\mathbb{O}}$ which is finite over $\text{Spec}(\mathbb{O})$. Also, the functor $\mathfrak{c}_2^{F_{\mathbb{O}}}$ is well-defined since we can choose a locally free resolution $0 \rightarrow F_1 \rightarrow F_0 \rightarrow F_{\mathbb{O}} \rightarrow 0$ and use the corresponding functors for F_1, F_0 (see next subsection for details). Then the restriction of $F_{\mathbb{O}}$ and $N = \mathfrak{c}_2^{F_{\mathbb{O}}} \otimes N_{\xi_{\mathbb{O}}}$ extends our quasibundle to $\text{Spec}(\mathbb{O})$. \square

Proposition 9 *If \mathbb{K} is a field and $\text{Spec}(\mathbb{K}) \rightarrow S$ is a morphism of schemes then the set of isomorphism classes $|QBun(r, n)(\text{Spec}(\mathbb{K}))|$ has a set-theoretic decomposition*

$$|QBun(r, d)(\text{Spec}(\mathbb{K}))| = \coprod_{d' \geq 0} |Bun(r, d - d')(\text{Spec}(\mathbb{K}))| \times \Gamma^{d'}(X)(\text{Spec}(\mathbb{K}))$$

Proof. A quasibundle over $\text{Spec}(\mathbb{K})$ always has the property that the vector bundle E on $U_{\mathbb{K}} \subset X_{\mathbb{K}}$ has a locally free direct image F . Applying Section 4.4 we see that a quasibundle in this case is simply a pair (F, ξ) where ξ is an effective cycle on $X_{\mathbb{K}}$ and F is a vector bundle. Since $\deg c_2(F) + \deg \xi = d$, the assertion follows. \square

5.3 Conjectured local covering by a scheme.

In this section we construct a scheme, or rather a system of schemes, mapping into the functor $QBun(r, d)$ and we conjecture that these give a faithfully flat covering for an increasing system of open substacks covering $QBun(r, d)$ (which would imply the $QBun(r, d)$ is an Artin Stack). We define an *enhanced quasibundle* as the following data

- A quasibundle (Z, E, N, D) as defined before;
- A section s of the line bundle D extending $\det(E)$ from U to X with zero scheme $Y \subset X$ being of pure relative dimension 1 over S and such that $Z \subset Y$;
- A line bundle L together with a finite dimensional subspace of section $H \subset H^0(X, L)$ which generate L . The projective space $P = \mathbb{P}(H)$ of lines in H has two open subsets: the subset \mathcal{U} of sections of L which are nonzero at the generic points of irreducible components of Y , and a smaller open subset \mathcal{U}_Z of those sections which are, in addition, nonzero at Z ;
- A vector bundle W on X pulled back from S and a quasimap $\mathcal{U} \rightarrow Quot_{Y \cap C}(W^\vee|_{Y \cap C}, l)$ where $l = L \cdot Y$ (the quasimap is understood in the sense of embedding the determinantal bundle on the $Quot$ scheme). We require that the line bundle on \mathcal{U} in the definition of a quasimap is

$$\Lambda^l \pi_* \mathcal{O}_{Y \cap C} \otimes N(L) \otimes \mathcal{O}_P(d)$$

- Over the open subset \mathcal{U}_Z , we require that the quasimap is a morphism (i.e. a map in the usual sense) and that we are given an isomorphism of E^\vee with the kernel of the composition $W^\vee \rightarrow W_Y^\vee \rightarrow Q$ where Q is the quotient sheaf corresponding to the morphism from \mathcal{U}_Z to the relative $Quot$ scheme.

For a fixed W and a relatively ample bundle B on X we conjecture that the subfunctor $QBun(r, d)_s$ of quasibundles such that $E \otimes B^{\otimes s}$ admits an enhancement for a choice of $s \geq 1$, form an increasing sequence of open subfunctors with the union equal to $QBun(r, d)$; and that the corresponding functors of enhanced quasibundles are represented by schemes which provide a faithfully flat cover of $QBun(r, d)_s$ for $s \geq 1$.

5.4 The Gieseker-to-Uhlenbeck morphism

Observe that our construction of the functor \mathfrak{c}_2 may be generalized a bit. Namely, suppose that E is a coherent sheaf, flat over S , which restricts to a torsion free sheaf on each fiber. Then at least locally on S we have a length 2 projective resolution $0 \rightarrow E_1 \rightarrow E_0 \rightarrow E \rightarrow 0$ in particular it is a perfect complex, so that $\det E$ is well-defined. Now we can use the same formula (3).

This defines a morphism of functors $Gies(r, d) \rightarrow QBun(r, d)$ where the source is the Gieseker compactification of flat families of torsion free sheaves with rank r and $c_2 = d$. Indeed, for any such sheaf E we can find $Z \subset X$ which is finite over S , such that E is locally free on the open complement $U = X \setminus Z$, and the multiplicative functor $N = \mathfrak{c}_2^E$ constructed above admits a natural E -localization at Z .

We need to show that this localization is effective. The latter condition can be checked on the fibers of π , i.e. we can assume that S is the spectrum of some field extension of k . But then $E^{\vee\vee}$ is locally free with $A = E^{\vee\vee}/E$ an Artin sheaf. Since

$$\mathfrak{c}_2^E \simeq \mathfrak{c}_2^{E^{\vee\vee}} \otimes N_A,$$

the assertion follows.

6 Simply-connected semi-simple groups.

6.1 Principal quasibundles on surfaces.

Let G is split semi-simple and connected group over k (automatically split since k is closed), with a maximal torus T , Weyl group W , character lattice \mathbb{X} and the dual lattice \mathbb{Y} . Denote also by $Q(G)$ the lattice of W -invariant symmetric bilinear forms on \mathbb{Y} . Recall that in our case such forms are automatically even and that $Q(G)$ is a free abelian group of rank t equal to the number of almost simple factors of G . We denote by $\pi(G)$ the dual free abelian group (it is isomorphic to the third homotopy group of G for $k = \mathbb{C}$). Both groups do not depend on the choice of T and a homomorphism $\rho : G_1 \rightarrow G_2$ of split semisimple simply connected groups induces a homomorphism $\rho_* : \pi(G_1) \rightarrow \pi(G_2)$. The meaning of $Q(G)$ is clarified by the following result, cf. Theorem 4.7 in [BD]:

Proposition 10 *Under the assumptions above $Q(Y) \simeq H_{Zar}^1(G, \mathcal{K}_2)$. Any \mathcal{K}_2 -torsor on G has a unique multiplicative structure and no nontrivial automorphisms.*

Observe also that the group $Q(G)$ contains a semigroup of non-negative quadratic forms, isomorphic to $(\mathbb{Z}^{\geq 0})^t$, and consequently $\pi(G)$ also has the dual semigroup $\pi_+(G)$ of non-negative elements. By the above proposition, on G we have a universal torsor over $\mathcal{K}_2 \otimes_{\mathbb{Z}} \pi(G)$ which we denote by \mathfrak{C}_2 .

Proposition 11 *A choice of a principal G -bundle P on X induces a multiplicative functor*

$$\mathfrak{c}_2^P : PIC(X/S) \rightarrow PIC(S) \otimes_{\mathbb{Z}} \pi(G)$$

where the target category is understood as torsors over $\mathcal{O}^ \otimes_{\mathbb{Z}} \pi(G) \simeq (\mathcal{O}^*)^{\times t}$. Moreover, this functor has degree d for some locally constant function $d : S \rightarrow \pi(G)$ (the generalized second Chern class of P) in the sense that for a line bundle L on S one has*

$$\mathfrak{c}_2^P(\pi^* L) \simeq L^{\otimes d}.$$

For a group homomorphism $\rho : G_1 \rightarrow G_2$ of semi-simple simply-connected groups one has a natural isomorphism of functors

$$\rho_* \circ \mathfrak{c}_2^P \simeq \mathfrak{c}_2^{\rho(P)}$$

where $\rho(P)$ is the induced principal bundle over G_2 .

Proof. First we give a definition assuming a relative Gersten conjecture for $\pi : X \rightarrow S$ and the sheaf \mathcal{K}_3 , and later we observe that the part of this conjecture which we really need to define the functor is actually known. For the sake of simplicity, we assume that P is locally free in Zariski topology. The relative version of the Gersten conjecture is a long exact sequence of sheaves in the Zariski topology on X :

$$0 \rightarrow \mathcal{K}_3 \rightarrow \mathcal{K}_3^{0/1}(X/S) \rightarrow \mathcal{K}_2^{1/2}(X/S) \rightarrow \mathcal{K}_1^{2/3}(X/S) \rightarrow 0;$$

plus the vanishing $R^i\pi_*$, $i > 0$ for the last three nonzero terms of the sequence.

Explicitly, \mathcal{K}_3 is sheafification in the Zariski topology, of algebraic K -theory for the category of projective modules on X , while $\mathcal{K}_j^{p/(p+1)}(X/S)$ is the sheafified K -theory for a certain category $\mathcal{W}_{X/S}^{p/(p+1)}$ of coherent sheaves which are flat over S and satisfy an additional property. It is required that a sheaf is supported at some zero scheme $X' \subset X$ for a relative regular sequence of length p over S , and locally over S has the form $M \otimes_{\mathcal{O}_{X'}} k(X'/S)$ where $k(X'/S)$ is the localization of $\mathcal{O}_{X'}$ at the multiplicative set of relative nonzero divisors; where the coherent sheaf M on X' is flat over S . See Theorem 4.8 in [Gr2] and definitions of that section for more details. Since $X \rightarrow S$ has relative dimension 2 for $\mathcal{K}_1^{2/3}(X/S)$ we are simply dealing with coherent sheaves which are flat over S and are supported at a zero scheme of a length 2 regular sequence. By Theorem 3.8 of [Gr1], $\mathcal{K}_3^{0/1}$ is the sheafification for the K_3 of $k(X/S)$, the “fiberwise rational functions on X ”.

Any G -torsor P on X induces a $\mathcal{K}_2 \otimes_{\mathbb{Z}} \pi(G)$ -gerbe $\mathfrak{C}_2(P)$ on X with a class in $H^2(X, \mathcal{K}_2 \otimes_{\mathbb{Z}} \pi(G))$ which we can call the second Chern class of P . These gerbe can either be understood as the gerbe of lifts of P to a torsor over the central extension \tilde{G} of G induced by \mathfrak{C}_2 ; or we can consider the canonical “cocycle map” $P \times_X P \rightarrow G$ and then pull back \mathfrak{C}_2 via this map. The result will be a “bundle gerbe”, cf. [St], which we also denote by $\mathfrak{C}_2(P)$, with the product structure on $P \times_X P \times_X P$ induced by the unique multiplicative structure of \mathfrak{C}_2 on G .

If we think of a line bundle L on X as a \mathcal{K}_1 -torsor L^* , then using the product $\mathcal{K}_1 \otimes_{\mathbb{Z}} \mathcal{K}_2 \rightarrow \mathcal{K}_3$ we can obtain a 2-gerbe on X . In the language of “bundle 2-gerbes” of [St] we first take $P^{[2]} = P \times_X P$ then set $Y \rightarrow P^{[2]}$ to be the pullback of L^* from X . On $Y^{[2]} = Y \times_{P^{[2]}} Y$ we have a \mathcal{K}_3 -torsor obtained by twisting the pullback of $\mathfrak{C}_2(P)$ with the cocycle on L .

By the relative Gersten resolution of \mathcal{K}_3 , we have $R^3\pi_*\mathcal{K}_3 = 0$ and therefore Zariski locally over S the 2-gerbe $\mathfrak{C}_2(P)$ can be trivialized. Comparing such trivializations on double intersections of an open covering for S , we get a Zariski torsor on S with the structure group $R^2\pi_*(\mathcal{K}_3 \otimes_{\mathbb{Z}} \pi(G))$.

$\pi(G)) \simeq \pi_*(\mathcal{K}_1^{2/3}(X/S) \otimes_{\mathbb{Z}} \pi(G))$ on S . Finally, to obtain a torsor over \mathcal{K}_1 on S , we need a norm map $\pi_*(\mathcal{K}_1^{2/3}(X/S) \otimes_{\mathbb{Z}} \pi(G)) \rightarrow \mathcal{K}_1$. Perhaps the easiest way to construct it is to use Nenashev's presentation of \mathcal{K}_1 via pairs of short exact sequences on the same triple of objects A, B, C , cf. [Ne]. In our case, the objects will be coherent sheaves on X which are flat over S and have finite support over S ; therefore we can simply consider the induced pair of short exact sequences on π_*A, π_*B, π_*C to obtain a \mathcal{K}_1 section on S .

All assertions of the proposition follow from definitions, if $d : S \rightarrow \pi(G)$ is obtained by applying the degree map $H^2(X(s), \mathcal{K}_2) \rightarrow \mathbb{Z}$ to the class of $\mathfrak{C}_2(P)$ in $H^2(X, \mathcal{K}_2)$, or using a similar resolution

$$0 \rightarrow \mathcal{K}_2 \rightarrow \mathcal{K}_2^{0/1}(X/S) \rightarrow \mathcal{K}_1^{1/2}(X/S) \rightarrow \mathcal{K}_0^{2/3}(X/S) \rightarrow 0. \quad \square$$

Remark. First we remark here on the assumption of Zariski local triviality of P . When $S = \text{Spec}(k)$, by the work of de Jong, He and Stuart, cf. [dJHS], any étale G torsor P has a rational section, and then by [Ni] it is actually Zariski locally trivial. It is natural to conjecture that for arbitrary S there is an étale covering $T \rightarrow S$ such that the pullback $P_T = P \times_S T$ is Zariski locally trivial. If this conjecture holds \mathfrak{C}_2^P would be constructed by étale descent (even in the case of algebraic spaces).

Next we discuss the features of the conjectural relative Gersten resolution which are important to our construction. Indeed, our complexes for \mathcal{K}_3 and \mathcal{K}_2 don't have to be exact in the first term, as we can always apply the map $\mathcal{K}_j \rightarrow \text{Ker}(\mathcal{K}_j^{0/1} \rightarrow \mathcal{K}_{j-1}^{1/2})$ to the gerbes in question. If $\mathcal{K}_j^p(X/S)$ for $p = 1, 2$ stands for the sheafified K-theory of coherent modules on X which are flat over S and have relative codimension p over S , then applying the argument on page 132 on [Gr1] we see that the induced map $\mathcal{K}_j^2(X/S) \rightarrow \mathcal{K}_j^1(X/S)$ is zero. Then the arguments of [Gr1] produce the complex of sheaves and show that it is exact except perhaps in the first term (keeping in mind that $\mathcal{K}_j^{2/3} = \mathcal{K}_j^2$ due to the relative dimension 2). As for the vanishing of $R^i \pi_* \mathcal{K}_j^{p/(p+1)}$ for $i > 0$ and $p = 0, 1, 2$ we use the fact that each sheaf $F = \mathcal{K}_j^{p/(p+1)}$ satisfies the relative version of the flasque condition (over S):

For any section $s \in F(U)$ and any point $x \in U$ there exists an open neighborhood V of $\pi(x) \in S$, such that $s|_{U \cap \pi^{-1}(V)}$ extends to $X_V = \pi^{-1}(V)$.

Then the classical proof of the fact that a flasque sheaf has trivial higher cohomology extends to the assertion that a sheaf F which is relatively flasque over S has trivial higher direct images (if π is quasicompact, as is the case in our setting). We are grateful to Piotr Achinger for explaining the last statement to us.

Although the sheaves $\mathcal{K}_j^{p/(p+1)}$ are not flasque in general, by the construction of K-theory groups in terms of acyclic binary multicomplexes, cf. [Gr3], it is fairly easy to see that these sheaves are indeed flasque over S , which gives the last feature that we need to construct the functor \mathfrak{C}_2^P and its degree function.

Remark. The same functor $\mathfrak{c}_2^P : PIC(X/S) \rightarrow PIC(S) \otimes_{\mathbb{Z}} \pi(G)$ may be interpreted as follows. Assume that L is a line bundle on X and its section s vanishes on a relative divisor $C \subset X$ which is a *smooth* family of curves over S . Then $P|_C$ defines a morphism of S to the relative moduli stack of G -bundles on C , and since $\pi(G)$ is dual to the Picard group of this moduli stack, cf. [BH], there is a pullback $\mathcal{O}^* \otimes_{\mathbb{Z}} \pi(G)$ torsor on S . In fact, we could attempt to *define* our functor by choosing an appropriate open covering of S and a section s_i over each open set of the covering. But then one has to show independence of the choice of s_i . Of course, another choice s'_i may be connected to s_j by a copy of \mathbb{P}^1 in the space of sections, but some of the sections in the family connecting s_i and s'_i will not give smooth curves over S . The best we can hope for is irreducible curves with nodal singularities. Since the corresponding torsors over the moduli of principal G bundles on nodal curves have not been constructed yet, we had to resort to the Gersten resolution approach as above.

Now the definition of a G -quasibundle on X follows the pattern similar to the case of vector bundles. More exactly, for a given locally constant degree function $d : S \rightarrow \pi(G)$ we consider a bundle P on the open complement $U \subset X$ to a closed subset Z which is finite over S , and a degree d multiplicative functor

$$N : PIC(X/S) \rightarrow PIC(S) \otimes_{\mathbb{Z}} \pi(G).$$

As in the case of vector bundles, we want to consider a P -localization of N at Z , i.e. for any section $s : \mathcal{O}_X \rightarrow L$ of a line bundle on X with zero scheme C disjoint from Z , we want to fix an isomorphism

$$a_s : \mathfrak{c}_2^P(L) \simeq N(L)$$

which behaves multiplicatively with respect to tensor products of line bundle and their sections. The LHS is well defined since P exists in an open neighborhood of C (in fact, we are even getting a gerbe over those sections of $\mathcal{K}_2^{1/2}$ which are supported on C , using that a P -bundle has a class in $H^2(C, \mathcal{K}_2)$).

Next, observe that any representation $\rho : G \rightarrow SL(r)$ induces a vector bundle $E = \rho(P)$, and a multiplicative functor $\rho(N) : PIC(X/S) \rightarrow PIC(S)$ of degree $\rho(d)$, which comes equipped with a natural E -localization.

Definition. We will say that a P -localization of N is *effective* if the induced $\rho(P)$ -localization of $\rho(N)$ is effective for any choice of a representation $\rho : G \rightarrow SL(r)$.

Remark. An alternative definition of an effective localization may be phrased as follows. Suppose that, after increasing Z , we can find a B -bundle P_0 inducing P on $X \setminus Z$ (where B is a Borel subgroup of G). Suppose further that the T -bundle induced from P_0 via $B \rightarrow B/U \simeq T$ does extend to a bundle on X which we also denote by P_1 .

Write the “universal symmetric W -invariant form” on \mathbb{Y} with values in $\pi(G)$ as an element of $\pi(G) \otimes_{\mathbb{Z}} \mathbb{X} \otimes_{\mathbb{Z}} \mathbb{X}$ in a certain basis x_1, x_2, \dots of \mathbb{X} :

$$\sum a_{ij} \otimes x_i \otimes x_j, \quad a_{ij} \in \pi(G).$$

Since each x_i is a character on T we can use it to produce a line bundle L_i from P_1 , which is defined everywhere on X . With this notation, set

$$N_{P_1} = \prod IB_{L_i, L_j}^{\otimes a_{ij}} : PIC(X/S) \rightarrow PIC(S) \otimes_{\mathbb{Z}} \pi(G).$$

One can show that, for our original P -localized N , the functor $N \otimes N_{P_1}^{-1}$ is \mathcal{O} -localized at Z , i.e. corresponds to a relative zero cycle over S but not necessarily effective. So we require that for every choice of a B -structure on P (perhaps after a base change on S and shrinking of an open set U on which P is defined), the quotient of the two multiplicative functors is given by an effective relative zero cycle. However, this definition is useful only if: (a) one proves a higher dimensional version of Drinfeld-Simpson lemma (which would state that after an etale base change a G -torsor admits a B -structure away from codimension two); (b) one has a useful technique for comparing different B -structures on the same G .

Definition. For a semi-simple simply connected G and a choice of d , a degree d *quasibundle* over G is defined as a triple (Z, P, N) consisting of a closed subset $Z \subset X$ which is finite over S , a principal G -bundle P on $U = X \setminus Z$, a degree d multiplicative functor $N : PIC(X/S) \rightarrow PIC(S) \otimes_{\mathbb{Z}} \pi_3(G)$; plus an effective P -localization of N at Z . We denote by $QBun(G, d)$ the functor of such quasibundles.

The properties listed below are either straightforward, or can be established by generalizing the techniques of [Ba3].

Proposition 12 *The functor $QBun(G, d)$ has the following properties:*

- A group homomorphism $\rho : G_1 \rightarrow G_2$ induces a morphism of functors $QBun(G_1, d) \rightarrow QBun(G_2, \rho_*(d))$;
- $QBun(G_1 \times G_2, d_1 \times d_2) \simeq QBun(G_1, d_1) \times QBun(G_2, d_2)$;
- There exists an injective group homomorphism $\rho : G \rightarrow \prod_i SL(r_i)$ such that the induced morphism of functors

$$QBun(G, d) \rightarrow \prod_i QBun(r_i, \rho_i(d))$$

is a closed embedding;

- The functor $Bun(G, d)$ of principal G -bundles with characteristic class d , is an open subfunctor of $QBun(G, d)$. The functor $QBun(G, d)$ is complete, i.e. for a DVR \mathbb{O} with the fraction field \mathbb{K} , a family of quasibundles over $Spec(\mathbb{K})$ extends (non-uniquely) to a family of quasibundles over $Spec(\mathbb{O})$.

- For a field \mathbb{K} with a morphism $\text{Spec}(\mathbb{K}) \rightarrow S$, the set of isomorphism classes of \mathbb{K} -valued points has a set-theoretic decomposition

$$|QBun(G, d)| = \coprod_{l \in \pi_+(G)} |Bun(G, d - l)| \times \Gamma^l(X)$$

where for $l = (l_1, \dots, l_t) \in (\mathbb{Z}^{\geq 0})^t \simeq \pi_+(G)$ we denote by

$$\Gamma^l(X) \simeq \Gamma^{l_1}(X) \times_S \dots \times_S \Gamma^{l_t}(X/S)$$

the space of $\pi_+(G)$ -effective 0-cycles on X .

Remark. In the case of general reductive G the situation is more delicate. See [BD] for the description of the category of multiplicative \mathcal{K}_2 -torsors on G and [BH] for the related description of the Picard group of the moduli stack of G -bundles on curves. We only note here that in defining quasibundles over this general G one needs to consider a G bundle P on the open subset U as before, but also fix an extension to X of the induced principal bundle over $G/[G, G]$ (e.g. the determinant line bundle for $G = GL(r)$).

6.2 Remarks on quasibundles in higher dimensions.

Let $\pi : X \rightarrow S$ be a smooth projective morphism with connected fibers of fixed dimension $e \geq 2$ (with some additional effort one could assume that X is only flat and projective over S , and satisfies Serre's condition S_2). A quasibundle on X is still defined as a bundle P over an open subset U with closed complement Z of codimension ≥ 2 over S , plus an appropriate multiplicative functor N with an effective P -localization at Z . The multiplicative functor N is now of the form

$$N : PIC(X/S)^{\times(e-1)} \rightarrow PIC(S) \otimes_{\mathbb{Z}} \pi(G)$$

Its degree is defined as a homomorphism

$$d : Pic(X/S)^{\times(e-2)} \rightarrow \underline{\pi(G)}$$

where $Pic(X/S)$ stands for the set of isomorphism classes in $PIC(X/S)$ and $\underline{\pi(G)}$ is understood as the sheaf of locally constant functions with values in $\pi(G)$. More explicitly, we require that

$$N(\pi^*L, L_1, \dots, L_{e-2}) \simeq L^{\otimes d(L_1, \dots, L_{e-2})},$$

and that N is symmetric in its arguments in the sense explained in [Du].

When P is defined everywhere on X , we could attempt to define such a functor such a functor \mathfrak{c}_2^P in two ways, but each has technical difficulties at the moment. First we can use the product

in K theory and multiply \mathcal{K}_1 -torsors L_0, \dots, L_{e-2} by the \mathcal{K}_2 -gerbe $\mathfrak{C}_2(P)$ to get an e -gerbe over \mathcal{K}_{e+1} , i.e. a geometric object having a class in $H^{e+1}(X, \mathcal{K}_{e+1})$. On each closed fiber over S such an e -gerbe can be trivialized, and all local trivializations induce an étale torsor over S with the structure group $R^e \pi_* \mathcal{K}_{e+1}$. Next, one needs a general relative Gersten resolution

$$0 \rightarrow \mathcal{K}_{e+1} \rightarrow \mathcal{K}_{e+1}^{0/1} \rightarrow \mathcal{K}_e^{1/2} \rightarrow \dots \rightarrow \mathcal{K}_1^{e/(e+1)} \rightarrow 0$$

with the property that the sheaves $\mathcal{K}_j^{(e+1-j)/(e+2-j)}$ have vanishing higher direct images $R^q \pi_*$, $q > 0$. In that case $R^e \pi_* \mathcal{K}_{e+1} \simeq \pi_* \mathcal{K}_1^{e/(e+1)}$ has a natural norm map to \mathcal{K}_1 (exactly as for $e = 2$) and one gets a \mathcal{K}_1 -torsor on S , which is the image of our functor. Here the technical difficulty is obviously constructing the relative Gersten resolution and proving the vanishing of higher direct images.

Alternatively, one can try to choose sections of L_0, \dots, L_{e-2} (locally over S) such that its common zero scheme $C \subset X$ is a family of smooth curves over S . Then S will map to the stack of G -bundles on the relative curve, and our functor is the pullback of the standard bundle on that stack. Here, as for the $e = 2$ case the difficulty is to show independence of the choice of sections, which requires existence of the required bundle for a family of curves with possible nodal singularities.

A general G -quasibundle will be, as before, a triple (Z, P, N) where $Z \subset X$ is a closed subset of pure relative dimension $(e - 2)$ over S , P a G -bundle over $U = X \setminus Z$, and N a multiplicative functor with $(e - 1)$ arguments, as above, *plus* an effective P -localization of N at Z , an identification of $N(L)$ and $\mathfrak{c}_2^P(L)$ for any section of L which is nonzero along Z . Effectiveness for principal bundles can be defined by looking at all induced vector bundles, and for a vector bundle E one requires that each short exact sequence $0 \rightarrow \mathcal{O}_X^{(r-1)} \rightarrow E \rightarrow \det(E) \rightarrow 0$ (which is chosen after a possible ample twist of E , base change on S and enlargement of Z), identifies the corresponding twist of N with the multiplicative functor of a codimension 2 *effective* cycle supported at Z . This depends on the fact that relative effective cycles can be described via similar multiplicative functors.

The product of zero cycle spaces $\Gamma^d(X/S)$ is now replaced by the Chow *scheme* $\text{Chow}^d(X/S)$ of multiplicative degree d functors with an appropriate \mathcal{O} -localization (which is a correct definition for the family of effective cycles in arbitrary characteristic). We expect that the functor $QBun(G, d)$ has a completeness property (i.e. a family over a punctured spectrum of a DVR extends - non-uniquely - to the puncture), and a decomposition for the set of points over $\text{Spec}(k)$:

$$QBun(G, d) = \coprod_{k \in \pi(G)} Bun'(G, d - k) \times \text{Chow}^k(X/S)$$

where $Bun'(G, \cdot)$ is the functor of G -bundles defined on an open subset of X with closed complement of codimension ≥ 3 . The latter is an Artin stack by [Ba3].

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